DUMMY PLAYERS AND THE QUOTA IN WEIGHTED VOTING GAMES*

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December 2019

Abstract. In a weighted voting game, each voter has a weight and a proposal is accepted if the sum of the weights of the voters in favor of that proposal is at least as large as a certain quota. It is well-known that, in this kind of voting process, it can occur that the vote of a player has no effect on the outcome of the game; such a player is called a "dummy" player. This paper studies the role of the quota on the occurrence of dummy players in weighted voting games. Assuming that every admissible weighted voting game is equally likely to occur, we compute the probability of having a player without voting power as a function of the quota for three, four and five players. It turns out that this probability is very sensitive to the choice of the quota and can be very high. The quota values that minimize (or maximize) the likelihood of dummy players are derived.[1]

JEL classification: **C7, D7**

Keywords: Cooperative game theory, weighted voting games, dummy player, probability of voting paradoxes.

*Financial support from the ANR SOLITER is gratefully acknowledged. Warm thanks are due to Sascha Kurz for his valuable help.

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[1]Some technical details are voluntarily omitted in this version of our study. These details can be found in the online appendix associated with this paper at https://bit.ly/2MVVuBW.

1 Introduction

In cooperative game theory, the power of a player in a voting game is defined as the probability to be decisive in the collective choice process. In a weighted voting game, if each player is given a weight that is both strictly positive and strictly lower than the quota (defined as the total weight needed to form a winning coalition), it could be expected that the voting power of every player is different from 0, that is that there is no *dummy* player. However, several (real) examples are given in the literature, showing that dummy players do exist. One of the most famous occurrences of a dummy player is offered by Luxembourg in the Council of Ministers of the EU between 1958 and 1973. Luxembourg held one vote, whereas the quota for a proposition to be approved was 12 out of 17. Since other member states held an even number of votes (4 for Germany, France and Italy, 2 for Belgium and The Netherlands), Luxembourg formally was never able to make any difference in the voting process and was a dummy. Such situations are obviously extremely undesirable but we should not worry about them if it could be shown that their occurrence is rare. Unfortunately, it is demonstrated in Barthélémy *et al.* (2013) that the theoretical probability of having a dummy player is far from being low, at least when the number of players is small. However, the study by Barthélémy *et al.* is restricted to *majority* voting games, in which 50% of the votes (weights) are sufficient for a proposition to be accepted. The purpose of the present study is to analyze more general weighted voting games and to investigate the impact of the quota value on the occurrence of a dummy player. Adopting a voting rule designer perspective, we search for the quota values that are susceptible to reduce the risk of a 'dummy paradox'. To achieve this goal, we compute in the current paper the probability to obtain a dummy player as a function of the quota and we determine the quota values which minimize (or maximize) this probability in weighted voting games with a small number of players.

To the best of our knowledge, the only related works come from the Artificial Intelligence literature. Zuckerman *et al.* (2012) study the effects that a change of the quota may have on a given player's power. They provide, in particular, an efficient algorithm for determining whether there is a value of the quota that makes a given player a dummy. The results they obtain demonstrate that even small changes in the quota can have a significant effect on a player's power. Some other results on the dependence between the players' powers and the quota in weighted voting games can be found in Zick *et al.* (2011), or in Boratyn *et al.* (2019). All these authors, however, adopt a general perspective, clearly different from our own point of view, which specifically focuses on the occurrence of dummy players.

The paper is built as follows: in the second section, our notation, definitions and assumptions are introduced. In the third section, we derive some analytical representations for the probability of having at least one dummy player in voting games with three, four and five players; these representations allow to calculate in each case the quota values that minimize (or maximize) the probability of having a dummy player. The case of six players will be only addressed through the above mentioned example of the European Union from 1958 to 1973. The main lessons of our study are summarized and discussed in the fourth section.

2 Notation, definitions and assumptions

A voting game is a pair (N, W) where N is the set of n players (or voters) and W the set of winning coalitions, that is the set of groups of players which can enforce their decision.[2]

In this paper, we consider the class of weighted voting games $[q; w_1, w_2, ..., w_n]$, where q is the quota needed to form a winning coalition and w_i is the number of votes (weight) of the ith player ; we assume that q and w_i are integers. A coalition S is winning if and only if $\sum_{i \in S} w_i \geq q$. The total number of votes, $\sum_{i \in N} w_i$, is denoted by w. A particular case is the majority game where $q_{maj} = \frac{w}{2} + 1$ if w is even and $q_{maj} = \frac{w+1}{2}$ if w is odd. We assume that the game is proper, that is $q \geq q_{maj}$. When $q = w$, we get the unanimity rule: each player has a veto power and is not a dummy. Our study will focus on the weighted voting games such that $q \leq w - 1$. The relative quota is denoted by Q with $Q = q/w$.

We assume, without loss of generality, that $w_1 \geq w_2 \geq ... \geq w_n \geq 0$.

Two weighted voting games with the same N are *isomorphic* (or equivalent) if they have the same set of winning coalitions. For example, $[5; 3, 2, 1]$ and $[2; 1, 1, 0]$ are isomorphic with $W = \{\{1, 2\}, \{1, 2, 3\}\}$. It is well-known that for the cases $n = 3$, 4 and 5, there exist, respectively, 8, 25 and 117 (classes of) weighted voting games (see, *e.g.*, Freixas and Molinero 2009).

A voter i is a *dummy* player in a voting game (N, W) if, for every coalition S, $S \in W$ implies $S \backslash \{i\} \in W$. In words, player i is never decisive in every winning coalition: the coalition wins with or without him (her). To illustrate, player 3 is a dummy in the weighted voting game $[5; 3, 2, 1]$. In voting power theory (see Straffin 1994; and Felsenthal and Machover 1998, for a presentation), it means that this player has no power.

A weighted voting game is said to be *admissible* if each player has at least one vote ($w_n \geq 1$) and never more than $q - 1$ votes (there is no dictator). We say that a *dummy paradox* occurs when an admissible weighted voting game is isomorphic to a weighted voting game with a least one dummy (*i.e.* one or more 0). Of course, if player j is a dummy player, then player k with $k > j$ is also a dummy. Also notice that, as $w_1 \leq q - 1$ in an admissible weighted voting game, player 2 is never a dummy and, consequently, the maximum number of dummy players is equal to $n - 2$.[3]

[2] The reader is referred to Taylor and Zwicker (1999) for a general presentation of voting games.

[3] In the particular case where $q = q_{maj}$, not only player 2 but also player 3 cannot be a dummy; see

The following notions will prove to be useful in Section 3. A winning coalition is called *minimal winning* if each proper subset is losing and a losing coalition is called *maximal losing* if each proper superset is winning. A minimal winning coalition S is called *shift-minimal winning* if no player j inside S can be replaced with a player i not in S, with $w_i < w_j$, such that the result is still winning; formally:

$$\forall j \in S, \forall i \in N \backslash S,\ w_i < w_j \Rightarrow (S \backslash \{j\}) \cup \{i\} \notin W.$$

Similarly, a maximal losing coalition S is called *shift-maximal losing* if:

$$\forall j \in S, \forall i \in N \backslash S,\ w_i > w_j \Rightarrow (S \backslash \{j\}) \cup \{i\} \in W.$$

In what follows, we will denote by $\mathcal{W}$ (respectively $\mathcal{L}$) the set of shift-minimal winning (shift-maximal losing) coalitions associated with a given weighted voting game. It can be noticed that a weighted voting game is totally characterized by $\mathcal{W}$ (or by $\mathcal{L}$).

The purpose of this paper is to compute the probability of obtaining a dummy player, given n, w and q (or Q), and to derive, when w tends to infinity, the quota which minimizes this probability (denoted by $\underline{Q}$) and the quota which maximizes this probability (denoted by $\overline{Q}$). The probability of having at least one dummy player is denoted by $P(n, w, q)$ or $P(n, w, Q)$ when w is finite and $P(n, Q)$ when w tends to infinity.

In order to compute $P(n, w, q)$ (or $P(n, w, Q)$ or $P(n, Q)$), we consider a particular probabilistic model - called IAC (Impartial Anonymous Culture) in voting theory - which is one of the most often used in such problems where the likelihood of a voting event is to be calculated (see, for instance, Moyouwou and Tchantcho 2015; Kamwa 2019; or Diss *et al.* 2018). In the current context, using this model consists in assuming that, n, w and q being given, all the admissible (integer) distributions of the w_i's, *i.e.* all the distributions such that ($q - 1 \geq w_1 \geq w_2 \geq ... \geq w_n \geq 1$, and $\sum_{i \in N} w_i = w$) are equally likely to occur. The choice of this model is based on computational considerations: in various contexts, a probability calculation under IAC is tantamount to compute the number of integer solutions of a set of linear inequalities with integer coefficients. There is a well established mathematical approach for performing such a calculation, based on Ehrhart's theory (Ehrhart 1977) and efficient counting algorithms.[4] We refer to Lepelley *et al.* (2008) and Wilson and Pritchard (2007) for more details on the use of these tools in probability calculations under IAC in voting theory. The probabilistic results presented in Section 3 have been obtained by applying (parameterized) Barvinok's algorithm (Barvinok 1994; Barvinok and Pommersheim 1999), implemented under [Barvinok] (2011).[5]

Proposition 1 in Barthélémy *et al.* (2013). In this case, the maximum number of dummy players is $n - 3$.

[4]For a general background on Ehrhart theory and on the general problem of counting integer points in polytopes, see for example Beck and Robins (2007).

[5]For a rigorous description of this algorithm and for implementation details, see Verdoolaege *et al.* (2004).

To illustrate our approach and our probabilistic assumption, suppose that $n = 3$ (three players) and $w = 7$ (the total number of votes is equal to 7). Assume first that the quota is $q = 4$. It is easy to check that, under our constraints, the only possible distributions of the votes between the three players are $(3, 3, 1)$ and $(3, 2, 2)$. The IAC model assumes that each of these two distributions occurs with a probability equal to 1/2. We note that with this quota (here $q = q_{maj}$), there is no dummy player: $P(3, 7, 4) = 0/2 = 0$ (in accordance with Corollary 1 in Barthélémy *et al.* 2013). Suppose now that $q = 5$. In this case, a further vote distribution is admissible: $(4, 2, 1)$; and each of the three vote distributions are equally likely to occur under IAC. Observe that player 3 is a dummy in the distribution $(3, 3, 1)$ and only in this distribution. Hence $P(3, 7, 5) = 1/3$. Assume finally that $q = 6$. In this case, four vote distributions are admissible, the three previous ones and $(5, 1, 1)$. As player 3 is a dummy in $(4, 2, 1)$ and in $(3, 3, 1)$ as well, we conclude that $P(3, 7, 6) = 2/4 = 1/2$.

3 Analytical representations

3.1 Three-player games

In this subsection, we derive probability representations for the three-player case. These representations are based on the following lemma.

Lemma 1 *In a three-player weighted voting game, a dummy player exists if and only if*

$$w_1 + w_3 \leq q - 1 \text{ and } w_1 + w_2 \geq q.$$

Proof. Player 3 is the only player who can be a dummy: if 2 and 3 are dummies, then player 1 is a dictator, in contradiction with our assumptions. Consider the possibly winning coalitions to which player 3 is susceptible to belong: $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. Player 3 is a dummy if either these coalitions are losing, or they are winning and they remain winning when player 3 is removed, *i.e.* if and only if: $(w_1 + w_3 \leq q - 1$ or $(w_1 + w_3 \geq q$ and $w_1 \geq q))$ and $(w_2 + w_3 \leq q - 1$ or $(w_2 + w_3 \geq q$ and $w_2 \geq q))$ and $(w_1 + w_2 + w_3 \leq q - 1$ or $(w_1 + w_2 + w_3 \geq q$ and $w_1 + w_2 \geq q))$. Given that (i) the grand coalition is always winning and (ii) a coalition with only one player cannot be winning (no dictator), these inequalities reduce to: $w_1 + w_3 \leq q - 1$, $w_2 + w_3 \leq q - 1$ and $w_1 + w_2 \geq q$. We obtain the desired result by noticing that $w_1 + w_3 \leq q - 1$ implies $w_2 + w_3 \leq q - 1$ (recall that $w_2 \leq w_1$). QED

Lemma 1 offers a very simple characterization of the vote distributions giving rise to a dummy player and makes possible the derivation of the following representation for $P(3, w, q)$.

Note also that these results could have been obtained by using the last version of the software [Normaliz] (2018), based on an algorithm different from Barvinok's one.

Proposition 1 *For $w = 3 \bmod 6$, the probability $P(3, w, q)$ is given as*

- *for $(w+3)/2 \leqslant q \leqslant 2w/3$:*

$$P(3,w,q) = -\frac{3(w^2 + 2w(1-2q) + (2q-1)^2)}{2w^2 - 6wq + 3(q^2-1)} \quad \textit{for } q \textit{ odd},$$

$$P(3,w,q) = -\frac{3(w^2 + 2w(1-2q) + (2q-1)^2)}{2w^2 - 6wq + 3(q^2-2)} \quad \textit{for } q \textit{ even};$$

- *for $(2w+3)/3 \leqslant q \leqslant w-1$:*

$$P(3,w,q) = \frac{3(5q - 3w - 2)(q-w)}{2w^2 - 6wq + 3(q^2-1)} \quad \textit{for } q \textit{ odd},$$

$$P(3,w,q) = \frac{3(3w^2 - 8wq + 2w + 5q^2 - 2q - 1)}{2w^2 - 6wq + 3(q^2-2)} \quad \textit{for } q \textit{ even}.$$

Proof. We compute first the total number of vote distributions in the three-player case. The parameters w and q being given (with $q_{maj} \leq q \leq w-1$), our constraints imply that a vote distribution is a vector of integers (w_1, w_2, w_3) such that:

$$w_1 \geq w_2,\ w_2 \geq w_3,\ w_3 \geq 1,\ w_1 \leq q-1, \text{ and } w_1 + w_2 + w_3 = w. \quad (1)$$

We know from Ehrhart's theory and its developments that the number of solutions of such a set of inequalities is a periodic quasi polynomial[6] in w and q, that we can obtain by using *e.g.* parametrized Barvinok's algorithm. In our case, it turns out that the quasi polynomial is as follows:

$$-\tfrac{1}{6}w^2 + \tfrac{1}{2}qw + [(-\tfrac{1}{4}q^2 + [0, \tfrac{1}{4}]_q), (-\tfrac{1}{4}q^2 + [\tfrac{1}{6}, -\tfrac{1}{12}]_q), (-\tfrac{1}{4}q^2 + [-\tfrac{1}{3}, -\tfrac{1}{12}]_q), (-\tfrac{1}{4}q^2 + [\tfrac{1}{2}, \tfrac{1}{4}]_q), (-\tfrac{1}{4}q^2 + [-\tfrac{1}{3}, -\tfrac{1}{12}]_q), (-\tfrac{1}{4}q^2 + [\tfrac{1}{6}, -\tfrac{1}{12}]_q)]_w. \quad (2)$$

Observe that the period that is associated with parameter w is equal to 6 and the period associated with$q$ is equal to 2. It means that the polynomial slightly differs depending on whether q is odd or even, and on whether w, $w+1$, $w+2$, $w+3$, $w+4$ or $w+5$ is a multiple of 6. For $w = 3 \bmod 6$, *i.e.* for $w+3$ multiple of 6 (the only case considered in Proposition 1), we obtain:

$$\begin{array}{l} -\frac{1}{6}w^2 + \frac{1}{2}qw - \frac{1}{4}q^2 + \frac{1}{2} \ \text{ if } q \text{ is even}, \\ -\frac{1}{6}w^2 + \frac{1}{2}qw - \frac{1}{4}q^2 + \frac{1}{4} \ \text{ if } q \text{ is odd}. \end{array}$$

[6] A quasi polynomial is a polynomial the coefficients of which are rational periodic numbers. Periodic numbers are usually made explicit by a list of rational numbers enclosed in square brackets. For example, $U(x) = [1/2, 3/4, 1]_x$ is a periodic number with period equal to 3 and with $U(x) = 1/2$ if $x = 0 \bmod 3$, $U(x) = 3/4$ if $x = 1 \bmod 3$ and $U(x) = 1$ if $x = 2 \bmod 3$.

Consider now the number of vote distributions for which a dummy exists. From Lemma 1, these vote distributions have to verify (in addition to conditions (1)):

$$w_1 + w_3 \leq q - 1 \text{ and } w_1 + w_2 \geq q. \quad (3)$$

Using Barvinok algorithm again, we obtain that the quasi polynomial giving the number of integer solutions of the sets of inequalities (1) and (3) differs depending on whether w is odd or even, but also on the value of q with respect to w. Two domains have to be distinguished.

For $(w+3)/2 \leq q \leq 2w/3$ (domain 1), we obtain:

$$\tfrac{1}{4}w^2 - qw + \tfrac{1}{2}w + [(q^2 - q + 0), (q^2 - q + \tfrac{1}{4})]_w, \qquad (4)$$

and for $(2w+3)/3 \leq q \leq w-1$ (domain 2), we have:

$$-\tfrac{3}{4}w^2 + 2qw - \tfrac{1}{2}w + [(-\tfrac{5}{4}q^2 + \tfrac{1}{2}q + [0, -\tfrac{1}{4}]_q), (-\tfrac{5}{4}q^2 + \tfrac{1}{2}q + [\tfrac{1}{4}, 0]_q)]_w \quad (5).$$

For $w = 3 \bmod 6$ and for $(w+3)/2 \leq q \leq 2w/3$, this reduces to

$$\frac{1}{4}w^2 - qw + \frac{1}{2}w + q^2 - q + \frac{1}{4},$$

and for $(2w+3)/3 \leq q \leq w-1$, we obtain:

$$-\frac{3}{4}w^2 + 2qw - \frac{1}{2}w - \frac{5}{4}q^2 + \frac{1}{2}q + \frac{1}{4} \text{ if } q \text{ is even,}$$

$$-\frac{3}{4}w^2 + 2qw - \frac{1}{2}w - \frac{5}{4}q^2 + \frac{1}{2}q \text{ if } q \text{ is odd.}$$

The desired representations for $w = 3 \bmod 6$ are then obtained by dividing the number of vote distributions with a dummy player by the total number of vote distributions. QED

Remark 1. Very similar representations dealing with the cases where w is different from 3 mod 6 can of course be obtained from relations (2), (4) and (5). This easy job is left to the reader (the resulting representations can be consulted in the online appendix). Some computed values of $P(3, w, Q)$, with $Q = q/w$, are listed in Table 1. These results show that the probability of having a dummy in the three-player case can reach very high values (close to 0.70); moreover, it turns out that $P(3, w, Q)$ first increases as the quota increases, then decreases for a quota higher than about 0.75.

Table 1. Numerical values of the probability of having a dummy with three players (in %)[7]

	Value of Q								
w	0.5	2/3	0.75	0.8	5/6	0.9	0.95	0.98	0.99
10	0	33.33	57.14	57.14	37.50	37.50	0	0	0
15	0	30.77	58.82	58.82	50.00	31.58	0	0	0
20	0	50.00	59.26	62.07	54.84	43.75	24.24	0	0
25	0	44.44	62.79	63.04	58.33	37.25	21.15	0	0
30	0	40.00	63.49	63.64	59.42	43.84	17.33	0	0
35	0	50.00	65.12	64.44	56.25	40.00	15.69	0	0
40	0	46.15	64.81	64.96	58.06	44.96	25.76	0	0
45	0	43.36	65.47	65.10	59.24	41.21	23.21	0	0
50	0	50.00	66.28	65.57	60.42	45.05	21.26	11.06	0
55	0	47.37	66.67	65.77	61.21	42.28	19.52	10.32	0
60	0	45.00	65.98	65.91	61.82	45.36	25.84	9.33	0
65	0	50.00	66.67	66.13	59.63	42.86	24.29	8.81	0
70	0	47.83	66.96	66.30	60.32	45.45	22.66	8.09	0
75	0	46.01	67.27	66.34	60.97	43.33	21.41	7.68	0
80	0	50.00	66.97	66.52	61.51	45.65	26.09	7.13	0
85	0	48.28	67.28	66.60	62.03	43.69	24.75	6.81	0
90	0	46.67	67.51	66.67	62.36	45.65	23.55	6.37	0
95	0	50.00	67.74	66.77	60.92	43.99	22.46	6.12	0
100	0	48.48	67.36	66.85	61.38	45.79	26.12	11.30	5.76
201	0	48.51	68.38	67.41	62.65	45.81	26.18	11.30	5.79
999	0	49.70	69.09	67.97	63.44	46.02	26.44	11.41	5.85

Remark 2. Let $P(3,Q)$ be the limiting probability of having a dummy player in a three-player weighted voting game when w tends to infinity. Simple representation for $P(3,Q)$ can easily be obtained from Proposition 1: replacing q with Qw (recall that $Q = q/w$) and making w tend to infinity, we get

$$P(3,Q) = \frac{-3(2Q-1)^2}{3Q^2-6Q+2} \quad \text{for } 1/2 \leqslant Q \leqslant 2/3,$$

$$P(3,Q) = \frac{3(Q-1)(5Q-3)}{3Q^2-6Q+2} \quad \text{for } 2/3 \leqslant Q \leqslant 1.$$

[7] When Qw is not an integer, we have computed the probability with q equal to the smallest integer higher than Qw.

From this representation, we verify that $P(3,Q)$ is minimized and equal to 0 when Q tends to 1 (unanimity) or to 1/2 (majority) and it is easy to obtain that $P(3,Q)$ is maximized for $\overline{Q} = 0.7676$, with $P(3,\overline{Q}) = 69.72\%$. Table 4 gives some calculated values of $P(3,Q)$.

Remark 3. Suppose that the values of Q are uniformly distributed on $[1/2,\ 1]$. It is easily obtained from our representation for $P(3,Q)$ that the *average* or *expected* probability of having a dummy player in 3-player weighted voting games is given as (for w tending to infinity)

$$E[P(3,Q)] = \frac{1}{3}\int_{1/2}^{2/3} 6P(3,Q)dQ + \frac{2}{3}\int_{2/3}^{1} 3P(3,Q)dQ = 40.83\%.$$

3.2 Four-player and five-player games

The four-player and five-player cases are more complex and we only propose analytical representations for the limiting probabilities, $P(4,Q)$ and $P(5,Q)$, assuming that w tends to infinity.

To obtain the inequalities that characterize the weight distributions associated with dummies for four or five players, it would be very tedious to proceed in the same way as for three players. We will base here our approach of the fact that weighted voting games has been classified for small numbers of players: exhaustive lists giving all possible (minimum integer representations for) non isomorphic voting games up to nine players exist in the literature (see e.g. Kurz 2012, and the numerous references given by this author).[8] We proceed as follows: first, we extract from the list associated witha given value of n the different types of weighted voting games with dummies : this corresponds to games whose minimum integer representation contains at least one w_i equal to zero; second we state, for each type of voting game with dummy(ies), the set of shift-minimal winning coalitions $\mathcal{W}$ and the set of shift-maximal losing coalitions $\mathcal{L}$; the set of characterizing inequalities is then given by $\sum_{i\in S} w_i \geq q$ for all $S \in \mathcal{W}$ and $\sum_{i\in T} w_i \leq q-1$ for all $T \in \mathcal{L}$. This approach is justified by the following two points: i) each game is isomorphic to its minimum integer representation, it therefore has the same set $\mathcal{W}$ and the same set $\mathcal{L}$ as its minimum representation; ii) each type of game is completely characterized by the constraints associated with its sets $\mathcal{W}$ and $\mathcal{L}$.

In the three-player case, there is only one type of voting game with dummy which reduces to: $[2;1,1,0]$. The unique shift-minimal winning coalition is $\{1,2\}$ and the unique

[8] We are very grateful to Sascha Kurz for having provided us with the exhaustive lists of minimum integer representations of weighted voting games up to seven players. 'Minimum integer' representation means that the weights are integers and every other integer representation is at least as large in each component. Note also that for $n \leq 7$, each weighted voting game admits one and only one minimum integer representation (see for example Freixas and Molinero 2009).

shift-maximal losing coalition is $\{1,3\}$. We then obtain: $w_1 + w_2 \geq q$ and $w_1 + w_3 \leq q - 1$, in accordance with Lemma 1.

For $n = 4$, 15 distinct weighted voting games are compatible with our constraints (no dictator, $q \geq q_{maj}$), and their minimum integer representations are: [5;3,2,2,1], [5;3,2,1,1], [5;2,2,1,1], [4,3,2,2,1], [4;3,1,1,1], [4;2,2,1,1], [4;2,1,1,1], [3;2,2,1,1], [3;2,1,1,1], [3;2,1,1,0], [3;1,1,1,1], [3;1,1,1,0], [2;1,1,1,1], [2;1,1,1,0], [2;1,1,0,0]. Among them, 4 voting games exhibit at least one dummy. These games are listed in the following Table, along with the associated shift-minimal winning and shift-maximal losing coalitions.

Table 2. List of four-player weighted voting games with dummies

Types of games	Shift-minimal winning coalitions	Shift-maximal losing coalitions
$[3;1,1,1,0]$	{1,2,3}	{1,2,4}
$[2;1,1,0,0]$	{1,2}	{1,3,4}
$[3;2,1,1,0]$	{1,3}	{1,4} and {2,3,4}
$[2;1,1,1,0]$	{2,3}	{1,4}

Lemma 2 immediately follows from Table 2.[9]

Lemma 2 *One (or two) dummy player(s) exist(s) in a four-player weighted voting game if and only if* ($w_1 + w_2 + w_3 \geq q$ *and* $w_1 + w_2 + w_4 \leq q - 1$) *or* ($w_1 + w_2 \geq q$ *and* $w_1 + w_3 + w_4 \leq q - 1$) *or* ($w_1 + w_3 \geq q$ *and* $w_2 + w_3 + w_4 \leq q - 1$ *and* $w_1 + w_4 \leq q - 1$) *or* $w_2 + w_3 \geq q$. *If* ($w_1 + w_2 \geq q$ *and* $w_1 + w_3 + w_4 \leq q - 1$), *and only in this case, two dummy players exist.*

Lemma 2 leads to the following proposition.

Proposition 2 *When* $n = 4$ *and* w *tends to infinity, the probability of obtaining at least one dummy player is*

$$P(4,Q) = \frac{2(-219Q^3 + 378Q^2 - 216Q + 41)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 1/2 \leqslant Q \leqslant 3/5,$$

$$P(4,Q) = \frac{2(156Q^3 - 297Q^2 + 189Q - 40)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 3/5 \leqslant Q \leqslant 2/3,$$

$$P(4,Q) = \frac{2(75Q^3 - 162Q^2 + 117Q - 28)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 2/3 \leqslant Q \leqslant 3/4,$$

$$P(4,Q) = \frac{2(-53Q^3 + 126Q^2 - 99Q + 26)}{4Q^3 - 12Q^2 + 12Q - 3} \quad for\ 3/4 \leqslant Q \leqslant 1;$$

[9]Notice that, as $q \geq q_{maj}$, we can omit the inequalities associated with $T \in \mathcal{L}$ when $N \backslash T$ is winning. This happens for the game $[2;1,1,1,0]$.

Proof. These results are obtained via the same approach as the one used to prove Proposition 1. But the representations we obtain (functions of both w and q) are unfortunately very complex and their practical use is limited. However, it is easy to deduce from these complicated expressions some limiting representations for the case where w tends to infinity (as already done in Remark 2): we have just to consider the highest degree term in w in the quasi polynomials we obtain.

Consider first the total number of vote distributions in a weighted voting game with four players. This number corresponds to the number of integer solutions of the following set of (in)equalities:

$$w_1 \geq w_2,\ w_2 \geq w_3,\ w_3 \geq w_4,\ w_4 \geq 1,\ w_1 < q \text{ and } w_1 + w_2 + w_3 + w_4 = w. \quad (6)$$

For $q_{maj} \leq q \leq w-1$, w odd, the quasi polynomial we obtain is as follows:

$$-\frac{1}{48}w^3 + (\frac{1}{12}q - \frac{1}{48})w^2 + (-\frac{1}{12}q^2 + \frac{1}{12}q + \frac{1}{48})w + \frac{1}{36}q^3 + f(q),$$

where $f(q)$ is a degree-2 quasi polynomial in q. Replacing q with Qn in this representation, we obtain

$$-\frac{1}{48}w^3 + (\frac{1}{12}wQ - \frac{1}{48})w^2 + (-\frac{1}{12}w^2Q^2 + \frac{1}{12}wQ + \frac{1}{48})w + \frac{1}{36}w^3Q^3 + f(wQ),$$

where the degree of w in $f(wQ)$ is lower than 3. The coefficient of w^3 (highest degree term in w) is:

$$-\frac{1}{48} + \frac{1}{12}Q - \frac{1}{12}Q^2 + \frac{1}{36}Q^3 = \frac{4Q^3 - 12Q^2 + 12Q - 3}{144}. \quad (7)$$

We consider now the number of vote distributions with at least a dummy player. In accordance with Lemma 2, we have to distinguish four cases for evaluating this number. Adding $w_1 + w_2 + w_3 \geq q$ and $w_1 + w_2 + w_4 \leq q - 1$ to (in)equalities (6) and proceeding as above, we find that the coefficient of w^3 in the quasi polynomial associated with the number of admissible weighted voting games isomorphic to $[3; 1, 1, 1, 0]$ is given as (notice that three domains have to be distinguished, depending on the value of Q):

$$0 \text{ for } Q \leq \frac{2}{3}$$

$$-\frac{2}{9} + Q - \frac{3}{2}Q^2 + \frac{3}{4}Q^3 \text{ for } \frac{2}{3} \leq Q \leq \frac{3}{4}$$

$$\text{and } \frac{19}{36} - 2Q + \frac{5}{2}Q^2 - \frac{37}{36}Q^3 \text{ for } \frac{3}{4} \leq Q \leq 1.$$

Similarly, for the number of admissible weighted voting games isomorphic to $[2; 1, 1, 0, 0]$, we find:

$$-\frac{1}{24} + \frac{1}{4}Q - \frac{1}{2}Q^2 + \frac{1}{3}Q^3 \text{ for } \frac{1}{2} \leq Q \leq \frac{2}{3}$$

$$\text{and} \quad -\frac{5}{24}+\frac{3}{4}Q-\frac{7}{8}Q^2+\frac{1}{3}Q^3 \quad \text{for} \quad \frac{2}{3}\leq Q\leq 1.$$

For $[3;2,1,1,0]$, we find:

$$\frac{1}{2}-\frac{11}{4}Q+5Q^2-3Q^3 \quad \text{for} \quad \frac{1}{2}\leq Q\leq \frac{3}{5}$$

$$-\frac{5}{8}+\frac{23}{8}Q-\frac{35}{8}Q^2+\frac{53}{24}Q^3 \quad \text{for} \quad \frac{3}{5}\leq Q\leq \frac{2}{3}$$

$$\text{and} \quad \frac{1}{24}-\frac{1}{8}Q+\frac{1}{8}Q^2-\frac{1}{24}Q^3 \quad \text{for} \quad \frac{2}{3}\leq Q\leq 1.$$

For the last type of weighted voting games with a dummy ($[2;1,1,1,0]$), it turns out that in this case the dummy paradox cannot occur for $Q > 2/3$ and we have:

$$\frac{1}{9}-\frac{1}{2}Q+\frac{3}{4}Q^2-\frac{3}{8}Q^3 \quad \text{for} \quad \frac{1}{2}\leq Q\leq \frac{2}{3}.$$

We obtain the representations in Proposition 2 by summing the above expressions for each appropriate value intervals for Q and dividing by (7). QED

Table 4 lists computed values of $P(4,Q)$ for various values of the relative quota Q. When the relative quota Q moves from $1/2$ to 1, the probability of having at least one dummy player decreases over the range of values with $1/2 \leq Q \leq 0.54$, then increases over the range $0.54 \leq Q \leq \overline{Q} = 0.8621$ and decreases again and tends to 0 when Q tends to 1. Notice that for $Q = 0.54$, $P(4,Q) = 32.81\%$ (local minimum) and for $Q = \overline{Q}$, we obtain $P(4,Q) = 68.49\%$, a surprisingly high value.

Consider now five-player games. Table 3 lists the 13 types of weighted voting games with dummies that can be encountered, together with the associated shift-minimal winning and shift-maximal losing coalitions.

Table 3. List of five-player weighted voting games with dummies

Types of games	Shift-minimal winning coalitions	Shift-maximal losing coalitions
$[4;1,1,1,1,0]$	{1,2,3,4}	{1,2,3,5}
$[3;1,1,1,0,0]$	{1,2,3}	{1,2,4,5}
$[5;2,2,1,1,0]$	{1,2,4}	{1,2,5} and {1,3,4,5}
$[2;1,1,0,0,0]$	{1,2}	{1,3,4,5}
$[5;3,2,1,1,0]$	{1,3,4} and {1,2}	{2,3,4,5} and {1,3,5}
$[4;2,2,1,1,0]$	{2,3,4} and {1,2}	{1,3,5}
$[4;2,1,1,1,0]$	{1,3,4}	{2,3,4,5} and {1,2,5}
$[3;2,1,1,0,0]$	{1,3}	{2,3,4,5} and {1,4,5}
$[5;3,2,2,1,0]$	{2,3,4} and {1,3}	{2,3,5} and {1,4,5}
$[4;3,1,1,1,0]$	{1,4}	{2,3,4,5} and {1,5}
$[3;2,1,1,1,0]$	{1,4} and {2,3,4}	{1,5} and {2,3,5}
$[3;1,1,1,1,0]$	{2,3,4}	{1,2,5}
$[2;1,1,1,0,0]$	{2,3}	{1,4,5}

Using exactly the same approach as above, this Table allows to state the needed set of characterizing inequalities for five players and to derive the following representations.

Proposition 3 *When $n = 5$ and w tends to infinity, the probability of obtaining at least one dummy player is*

$$P(5,Q) = -\frac{5(35383Q^4 - 78168Q^3 + 64728Q^2 - 23808Q + 3282)}{6(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 1/2 \le Q \le 5/9$$

$$P(5,Q) = \frac{5(3983Q^4 - 9312Q^3 + 8172Q^2 - 3192Q + 468)}{6(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 5/9 \le Q \le 4/7$$

$$P(5,Q) = \frac{5(791Q^4 - 1912Q^3 + 1734Q^2 - 700Q + 106)}{3(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 4/7 \le Q \le 3/5$$

$$P(5,Q) = \frac{5(17457Q^4 - 42524Q^3 + 38838Q^2 - 15764Q + 2399)}{6(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 3/5 \le Q \le 5/8$$

$$P(5,Q) = -\frac{5(31695Q^4 - 80356Q^3 + 76362Q^2 - 32236Q + 5101)}{6(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 5/8 \le Q \le 2/3$$

$$P(5,Q) = \frac{5(14313Q^4 - 40388Q^3 + 42726Q^2 - 20084Q + 3539)}{6(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 2/3 \le Q \le 5/7$$

$$P(5,Q) = -\frac{5(3648Q^4 - 10676Q^3 + 11712Q^2 - 5708Q + 1043)}{3(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 5/7 \le Q \le 3/4$$

$$P(5,Q) = -\frac{5(1088Q^4 - 3380Q^3 + 3936Q^2 - 2036Q + 395)}{3(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 3/4 \le Q \le 4/5$$

$$P(5,Q) = \frac{5(787Q^4 - 2620Q^3 + 3264Q^2 - 1804Q + 373)}{3(5Q^4 - 20Q^3 + 30Q^2 - 20Q + 4)} \quad \textit{for } 4/5 \le Q \le 1;$$

The proof follows the same line as the one of Proposition 2 and is omitted (some details are however available on the online appendix associated with this paper).

Computed values of $P(5,Q)$ are displayed in Table 4. It turns out that $P(5,Q)$ is minimized for $\underline{Q} = 0.5234$ with $P(5,\underline{Q}) = 35.42\%$ and maximized for $\overline{Q} = 0.9131$ with $P(5,\overline{Q}) = 65.50\%$. Assuming that Q is uniformly distributed on $[1/2, 1]$, we obtain the following *average* probability: $E[P(5,Q)] = 50.24\%$.

Figure 1 represents the limiting probabilities according to the quota Q for three, four and five players. Contrary to the three-player case, the cases with four and five players do not lead to a regular single-peaked function:[10] as mentioned above, in addition to the global minimum reached for $Q = 1$, we obtain for $n = 4$ and $n = 5$ a local minimum which is not far from the majority case.

Table 4. Numerical values of the limiting probability of having at least one dummy for three, four and five players (in%)

Quota Q	$P(3,Q)$	$P(4,Q)$	$P(5,Q)$
1/2	0	50.00	53.03
0.51	0.29	40.51	39.98
0.52	1.55	35.17	35.33
0.55	7.64	34.26	40.25
0.60	23.08	47.31	45.56
0.65	42.69	50.45	48.25
2/3	50.00	52.17	49.34
0.70	61.64	54.93	51.86
0.75	69.23	56.67	53.12
0.80	68.18	62.81	54.57
0.85	60.32	68.20	58.25
0.90	46.39	64.86	64.91
0.95	26.45	44.85	57.69
0.98	11.41	21.45	32.33
0.99	5.85	11.35	17.82
1	0	0	0

[10]The particular shape of the curve for the 3-player case is strongly impacted by the fact that the probability of having a dummy player is, in this case, equal to zero when $Q = 1/2$.

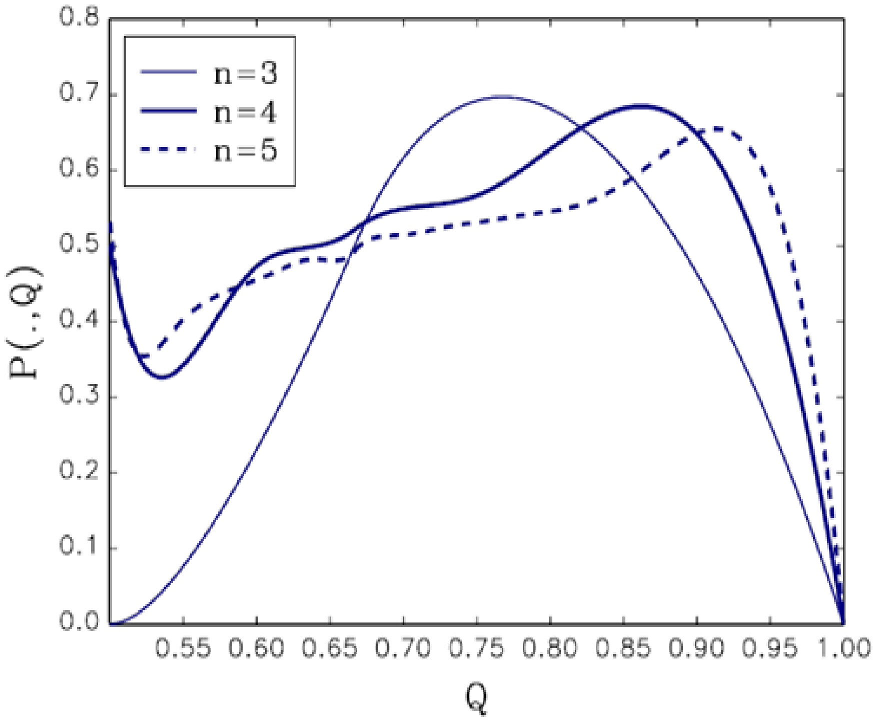

Figure 1: Limiting probability of having at least one dummy player for three, four and five players

3.3 Six-player games: the case of EU6

For six players, the number of distinct weighted voting games with dummies that we have to consider is equal to 62 and makes the computations, if not impossible, at least very tedious. For this reason, we have to resort to other methods (simulations) to know what happens when the number of players is higher than five. These simulations are beyond the scope of this preliminary study. It is however possible to get some insight for the case of European Union with six members (denoted by EU6 thereafter), already mentioned in our Introduction. The weighted voting game chosen by the six member states in 1958 was: $[12; 4, 4, 4, 2, 2, 1]$. We would like to answer the following question: what was the likelihood of a dummy paradox, given the leading principles behind the choice of this specific voting game? It can be suggested that the six countries agreed on some rules that may be formulated as follows: the three 'big' countries - Germany, France and Italy - should have the same weights ($w_1 = w_2 = w_3$), the two medium countries - Belgium and Netherlands - should also have the same weight ($w_4 = w_5$) and it seemed fair to have $w_3 \geq w_4$ and $w_5 \geq w_6 \geq 1$, with Luxemburg as sixth player. In addition, the choice of the quota suggests that it was desired that the three big countries should be decisive when voting together (*i.e.* $3w_1 \geq q$). We say that an admissible weighted voting game is of EU6 type if it verifies these additional constraints. We have computed the proportion of EU6 voting games with at least one dummy as function of Q. The computation of the number of EU6 voting games with dummy is based on the following Table:

Table 5. List of EU6 weighted voting games with dummies

Types of games	Shift-minimal winning coalitions	Shift-maximal losing coalitions
$[3; 1, 1, 1, 0, 0, 0]$	{1,2,3}	{1,3,4,5,6}
$[6; 2, 2, 2, 1, 1, 0]$	{1,2,3} and {2,3,4,5}	{1,2,4,6}
$[5; 2, 2, 2, 1, 1, 0]$	{2,3,5}	{1,2,6} and {1,4,5,6}
$[2; 1, 1, 1, 0, 0, 0]$	{2,3}	{1,4,5,6}
$[3; 1, 1, 1, 1, 1, 0]$	{3,4,5}	{1,2,6}

Let $P(EU6, Q)$ denote the probability of having at least one dummy in EU6 voting games. We obtain (see online appendix for details):

$$P(EU6, Q) = -\frac{555Q^2 - 600Q + 164}{16(5Q^2 - 5Q + 1)} \quad \text{for } 1/2 \leq Q \leqslant 5/9,$$

$$P(EU6, Q) = \frac{660Q^2 - 750Q + 211}{16(5Q^2 - 5Q + 1)} \quad \text{for } 5/9 \leq Q \leqslant 3/5,$$

$$P(EU6, Q) = \frac{163Q^2 - 204Q + 64}{4(Q - 1)^2} \quad \text{for } 3/5 \leq Q \leqslant 2/3,$$

$$P(EU6,Q) = \frac{25Q^2 - 36Q + 13}{(Q-1)^2} \quad \text{for } 2/3 \leq Q \leqslant 3/4,$$
$$P(EU6,Q) = 1 \quad \text{for } 3/4 \leq Q < 1.$$

The expected probability is approximately given as $E[P(EU6,Q)] = 78\%$ and we conclude that EU6 was not particularly unlucky by getting a dummy: it was a fairly normal thing given the principles underlying the determination of the game rule! We can even think that EU6 was actually rather lucky: under our assumptions - admittedly disputable -, it turns out (see appendix) that the average probability of having three dummies (Belgium, Netherlands and Luxembourg) was equal to 58.5%!

4 Conclusion and final remarks

The main lessons that emerge from the above results can be summarized as follows.

1) Barthélémy *et al.* (2013) have shown that the probability of having a dummy player is surprisingly high in majority voting games ($Q = 50\%$). Our results demonstrate that increasing the quota does not reduce in most of the cases the probability that the 'dummy paradox' occurs. In the three-player case, this (limiting) probability is maximized for $\overline{Q} = 0.77$ and the corresponding quota values seem to increase for $n > 3$ since we find $\overline{Q} = 0.86$ for $n = 4$ and $\overline{Q} = 0.91$ for $n = 5$.

2) In order to minimize the probability of having dummy players, it is advisable to choose a quota that is not too high, at least when the choice is restricted to "standard" quotas (1/2, 3/5, 2/3, 3/4, 4/5...): in this case, taking $Q = 1/2$ for $n = 3$ and $Q = 3/5$ for $n = 4$ and $n = 5$ appear to be the right solution. The choice of a quota close to 1 (*e.g.* 0.95) that could be suggested by the observation that there is no dummy player for $Q = 1$ would be a serious mistake for $n = 5$.

Remark 4. Our results suppose that player 1, the 'biggest' player, holds a weight that may be (almost) as high as the quota: $w_1 \leq q - 1$. This assumptions is of course disputable and it seems of interest to study the extent to which it impacts our results. In order to clarify this question, we have considered (for the 3 and 4-player cases) the more constrained -but perhaps more realistic- situation where the number of votes of the biggest player may not be higher or equal to half of the total number of votes: $w_1 < w/2$. Let $P^*(n,Q)$ be the probability of having at least one dummy player under this stronger constraint when w tends to infinity. We only consider here the cases with $n = 3$ and $n = 4$.

Using the same approach as in Section 3, we obtain the following results for the three-player case (the details regarding all the results given in this Remark can be found in the online appendix):

$$P^*(3,Q) = 6(2Q-1)^2 \quad \text{for } 1/2 \leq Q \leqslant 2/3,$$

$$P^*(3,Q) = -6(14Q^2 - 20Q + 7) \text{ for } 2/3 \leqslant Q \leqslant 3/4,$$

$$P^*(3,Q) = 12(Q-1)^2 \text{ for } 3/4 \leqslant Q \leq 1.$$

The comparison of the computed values of $P^*(3,Q)$ (not reported here) with the values listed in Table 4 shows that $P^*(3,Q)$ is lower than $P(3,Q)$ for values of Q close to 1/2 or 1; however, we observe that for intermediate values of Q, $P^*(3,Q)$ is clearly higher than $P(3,Q)$. On the average, we obtain that $E[P^*(3,Q)] = 1/3 < E[P(3,Q)] = 40.83\%$: introducing the additional constraint that $w_1 < w/2$ tends to reduce the risk of having a dummy player but this risk still remains high.

In the four-player case, the results are the following :

$$P^*(4,Q) = 2(2-3Q)(126Q^2 - 132Q + 35) \quad \text{for } 1/2 \leq Q \leqslant 3/5,$$

$$P^*(4,Q) = 2(372Q^3 - 702Q^2 + 441Q - 92) \quad \text{for } 3/5 \leq Q \leqslant 2/3,$$

$$P^*(4,Q) = 2(48Q^3 - 108Q^2 + 81Q - 20) \quad \text{for } 2/3 \leq Q \leqslant 3/4,$$

$$P^*(4,Q) = -(512Q^3 - 1224Q^2 + 972Q - 257) \quad \text{for } 3/4 \leq Q \leqslant 5/6,$$

$$P^*(4,Q) = 2(Q-1)(68Q^2 - 130Q + 59) \quad \text{for } 5/6 \leq Q \leq 1.$$

From computed values of $P^*(4,Q)$, we conclude in the same way as for the three-player case: requiring that w_1 must not be higher or equal to $w/2$ reduces the probability of having a dummy for some values of Q but increases this probability for other values. We obtain that the expected probability is given as $E[P^*(4,Q)] = 47.08\%$, a value slightly lower than $E[P(4,Q)] = 50.97\%$. An interesting observation is that our alternative assumption tends to reduce the probability of dummy players for high values of the quota when $n = 3$ and $n = 4$.

Remark 5. Two key issues remain regarding our conclusions: first, we only consider in this paper a small number of players; second, our results are based on a very specific probability model (IAC), that could exaggerate the probabilities of having a dummy player. These two issues are addressed in a companion paper (Barthélémy and Martin 2019), to which the interested reader may refer. The results presented in this paper tend to confirm and allow to specify our overall conclusions.

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